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A TANGENTIAL CONVERGENCE FOR BOUNDED HARMONIC FUNCTIONS ON A RANK ONE SYMMETRIC SPACE

BY

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ABSTRACT. Let u be a bounded harmonic function on a noncompact rank one symmetric space $M = G/K \approx N^-A$, N^-AK being a fixed Iwasawa decomposition of G. We prove that if for an $a_0 \in A$ there exists a limit $\lim u(na_0) \equiv c_0$, as $n \in N^-$ goes to infinity, then for any $a \in A$, $\lim u(na) = c_0$. For $M = SU(n, 1)/S(U(n) \times U(1)) = B^n$, the unit ball in $\mathbf{C}^n$ with the Bergman metric, this is a result of Hulanicki and Ricci, and in this case it reads (via the Cayley transformation) as a theorem on convergence of a bounded harmonic function to a boundary value at a fixed boundary point, along appropriate, tangent to ∂B^n, surfaces.

0. Introduction. Let M be a noncompact symmetric space of rank one. M can be expressed as a homogeneous space G/K where G is a semisimple group of isometries of M and K is a maximal compact subgroup of G. Let $\mathfrak{g}$, $\mathfrak{k}$ denote the Lie algebras of G and K, B the Killing form of $\mathfrak{g}$, and $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to B. If π: $G \to G/K$ denotes the canonical projection, its differential at the identity, π_*, identifies the subspace $\mathfrak{p}$ of $\mathfrak{g}$ with $T_0(M)$, the tangent space of M at the origin $o = \pi(e)$, and the invariant metric g on M can be chosen so that g_0 corresponds to the restriction of B to $\mathfrak{p} \times \mathfrak{p}$ under the above identification. We denote by Δ the corresponding (G-invariant) Laplace-Beltrami operator on M. A function $u \in C^\infty(M)$ is called *harmonic* if $\Delta u = 0$. Let $\mathfrak{a}$ be a maximal (one-dimensional) abelian subspace of $\mathfrak{p}$, α and possibly 2α in $\mathfrak{a}^*$, the corresponding system of positive restricted roots relative to the fixed choice of a "positive part" $\mathfrak{a}^+$ in $\mathfrak{a}$. Let $\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$ denote the root spaces corresponding to $-\alpha$ and -2α. Then $\mathfrak{n}^- = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$ is a nilpotent subalgebra of $\mathfrak{g}$ and one has the Iwasawa decomposition $G = N^-AK$, with $N^- = \exp \mathfrak{n}^-$, $A = \exp \mathfrak{a}$. The above decomposition shows that every $p \in M$ can be uniquely written as $p = na \cdot o$ ($n \in N^-$, $a \in A$). We regard the nilpotent group N^- as a boundary for the symmetric space M in the following sense. The bounded harmonic functions u on M have boundary values on N^-, i.e. $\lim_{\log a \to \infty} u(na \cdot o) \equiv \varphi(n)$ exists a.e. (relative to the Haar measure on N^-) and $\varphi \in L^\infty(N^-)$. $\log a \to \infty$ is understood with respect to the ordering induced on $\mathfrak{a}$ by $\mathfrak{a}^+$. Moreover,

$$u(na \cdot o) = \varphi * P_a(n) = \int_{N^-} \varphi(n_1) P_a(n_1^{-1}n)\, dn_1.$$

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The function $P_a(n)$ on $N^- \times A$ is called the Poisson kernel for the symmetric space M and is given by (Helgason [4])

$$P_a(n) = c\varepsilon^{d/2}\Big[\big(\varepsilon + \tfrac{1}{2}Q(X_{-\alpha})\big)^2 + 2Q(X_{-2\alpha})\Big]^{-d/2},$$

where

$$n = \exp(X_{-\alpha} + X_{-2\alpha}), \qquad X_{-\alpha} \in \mathfrak{g}_{-\alpha}, \ X_{-2\alpha} \in \mathfrak{g}_{-2\alpha};$$
$$\varepsilon = e^{-\alpha(\log a)}; \qquad Q(X) = (X, X)_\theta / 2(m_\alpha + 4m_{2\alpha})$$

with $(X, X)_\theta = -B(X, \theta X)$ for $X \in \mathfrak{g}$, θ denoting the Cartan involution associated with the pair $(\mathfrak{g}, \mathfrak{k})$; $m_\alpha = \dim \mathfrak{g}_{-\alpha}$, $m_{2\alpha} = \dim \mathfrak{g}_{-2\alpha}$, $d = m_\alpha + 2m_{2\alpha}$. The constant c is such that the integral of P_a over N^- is equal to 1.

The following theorem on "tangential" convergence for bounded harmonic functions on the Siegel domain

$$D_{r-1} = \left\{(z_1, \ldots, z_r) \in \mathbf{C}^r\colon \operatorname{Im} z_r > \sum_{j=1}^{r-1} |z_j|^2\right\},$$

$r \geqslant 2$, (or, equivalently, on $M = SU(r, 1)/S(U(r) \times U(1))$–the complex hyperbolic space) has been obtained by Hulanicki and Ricci [5]. We formulate it below in terms of a homogeneous space M.

THEOREM. *Let u be a bounded harmonic function on a noncompact rank one symmetric space M. In the notation above, assume that for an $a_0 \in A$, $\lim_{N^- \ni n \to \infty} u(na_0 \cdot o) = c_0$. Then for any $a \in A$, $\lim_{N^- \ni n \to \infty} u(na \cdot o) = c_0$.*

Our aim here is to prove the above Theorem and the proof is based on the classification of symmetric spaces. That is, we discuss separately the cases of M being the real, complex (to see how the $M = D_{r-1}$ case fits to our scheme), quaternion and octonion hyperbolic space, which corresponds respectively to G being the classical group $SO_0(r, 1)$, $SU(r, 1)$, $Sp(r, 1)$ and the exceptional group $F_{4(-20)}$. Following the Hulanicki-Ricci method, for each case we construct a suitable commutative subalgebra $\mathcal{Q}$ of (multi) radial functions in $L^1(N^-)$, to which the Poisson kernel P_a belongs. We describe the set $\mathfrak{M}(\mathcal{Q})$ of the maximal ideals in $\mathcal{Q}$ and check that the Gel'fand transform $\hat{P}_a$ of P_a never vanishes on $\mathfrak{M}(\mathcal{Q})$. The Theorem may then be stated as a theorem on certain ideals in $L^1(N^-)$ and is a consequence of the Wiener property of the algebra $\mathcal{Q}$. To study the algebra $\mathcal{Q}$ we use the holomorphically induced (realizations of the irreducible unitary) representations of N^-.

1. Nilpotent group N^-. Let $\mathbf{F}$ denote the field $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ or the Cayley numbers $\mathbf{O}$ (octonions); $\mathbf{F}_0 = \{q \in \mathbf{F}\colon q + \bar{q} = 0\}$, $^-$ being the usual conjugation in $\mathbf{F} = \mathbf{C}, \mathbf{H}, \mathbf{O}$ and $\bar{q} = q$ for $q \in \mathbf{R}$; $\operatorname{Im} q = \frac{1}{2}(q - \bar{q})$, $\sigma = 2s = \dim_{\mathbf{R}} \mathbf{F}$. According to the notation of the previous section, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and for the classical G we have (cf., e.g., [3, pp. 348–351])

$$\mathfrak{k} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & p \end{pmatrix} : \begin{array}{c} Z \text{ an } r \times r \text{ skew-Hermitian matrix over } \mathbf{F}, \\ p \in \mathbf{F}_0, \operatorname{tr} Z = -p \text{ in case of } \mathbf{F} = \mathbf{C} \end{array} \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & {}^t q \\ \bar{q} & 0 \end{pmatrix} : q \in \mathbf{F}^r = \mathbf{F} \times \cdots \times \mathbf{F} \right\},$$

$$\mathfrak{a} = \{ tE_{1,r+1} + tE_{r+1,1} : t \in \mathbf{R} \},$$

where E_{kl} denotes the $(r+1) \times (r+1)$ matrix $(\delta_{ak}\delta_{bl})_{1 \leqslant a,b \leqslant r+1}$, $r \geqslant 2$. We choose a basis $H = E_{1,r+1} + E_{r+1,1}$ in $\mathfrak{a}$ and fix an ordering so that $H \in \mathfrak{a}^+$. Then $\alpha \in \mathfrak{a}^*$ such that $\alpha(H) = 1$ is a positive restricted root, and we have

$$\mathfrak{g}_{-\alpha} = \left\{ \begin{bmatrix} 0 & -\bar{q} & 0 \\ {}^t q & 0 & {}^t q \\ 0 & \bar{q} & 0 \end{bmatrix} : q = (q_2, \dots, q_r) \in \mathbf{F}^{r-1} \right\},$$

$$\mathfrak{g}_{-2\alpha} = \left\{ \begin{bmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & 0 & \bar{p} \end{bmatrix} : p \in \mathbf{F}_0 \right\} \quad (= \{0\} \text{ for } \mathbf{F} = \mathbf{R}).$$

We shall identify $\mathfrak{n}^- = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$ with $\mathbf{F}^{r-1} \times \mathbf{F}_0$ by the correspondence

$$\begin{bmatrix} p & -\bar{q} & p \\ {}^t q & 0 & {}^t q \\ \bar{p} & \bar{q} & \bar{p} \end{bmatrix} \leftrightarrow (q, -p).$$

In these coordinates on $\mathfrak{n}^-$ the commutator of $(q, p) = (q_1, \dots, q_{r-1}, p)$ and $(q', p') = (q'_1, \dots, q'_{r-1}, p')$ in $\mathbf{F}^{r-1} \times \mathbf{F}_0$ is given by

$$[(q,p),(q',p')] = (0, 2\operatorname{Im}(\bar{q} \cdot q')), \tag{1}$$

where we have put $\bar{q} \cdot q'$ for $\sum_{l=1}^{r-1} \bar{q}_l q'_l$. We also have the formula (cf., e.g., [**11**, p. 39])

$$((q,p),(q',p'))_\theta = 4(m_\alpha + 4m_{2\alpha})\operatorname{Re}(\bar{q} \cdot q' + \bar{p}p'). \tag{2}$$

For the exceptional G (cf., e.g., [**10**, pp. 522–530]), $\mathfrak{g} = \mathfrak{f}_{4(-20)}$ is isomorphic to the Lie algebra $\operatorname{Der}(\mathfrak{J})$ of derivations of the Jordan algebra $(\mathfrak{J}, \circ)$ of 3×3 octonion matrices A of the form

$$A = \begin{bmatrix} \alpha_1 & a_3 & a_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ -\bar{a}_2 & -\bar{a}_1 & \alpha_3 \end{bmatrix}, \qquad a_i \in \mathbf{O}, \quad \alpha_i \in \mathbf{R}, \qquad i = 1, 2, 3,$$

with multiplication given by $A \circ B = \frac{1}{2}(AB + BA)$, $A, B \in \mathfrak{J}$, AB denoting the usual matrix multiplication. We have

$$\mathfrak{k} = \{ D \in \operatorname{Der}(\mathfrak{J}) : D(E_{33}) = 0 \},$$

$$\mathfrak{p} = \left\{ D_Q \in \operatorname{Der}(\mathfrak{J}) : Q = \begin{pmatrix} 0 & {}^t q \\ \bar{q} & 0 \end{pmatrix}, \quad q \in \mathbf{O}^2 \right\},$$

where $D_Q(B) = QB - BQ$, $B \in \mathfrak{J}$.

$$\mathfrak{a} = \{D_Q \in \mathrm{Der}(\mathcal{J}) \colon Q = tE_{13} + tE_{31}, \quad t \in \mathbf{R}\}.$$

We choose $H = D_Q$ with $Q = E_{13} + E_{31} \in \mathfrak{a}^+$ and $\alpha \in \mathfrak{a}^*$ such that $\alpha(H) = 1$. Then

$$\mathfrak{g}_{-\alpha} = \left\{ D_{Q(q)} \colon Q(q) = \begin{bmatrix} 0 & -\bar{q} & 0 \\ q & 0 & q \\ 0 & \bar{q} & 0 \end{bmatrix}, \quad q \in \mathbf{O} \right\},$$

$$\mathfrak{g}_{-2\alpha} = \left\{ D_{Q(p)} \colon Q(p) = \begin{bmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & 0 & \bar{p} \end{bmatrix}, \quad p \in \mathbf{O}_0 \right\}.$$

Identifying $D_{Q(q)} + D_{Q(p)}$ in $\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$ with $(q, -p)$ in $\mathbf{O} \times \mathbf{O}_0$, we obtain the same formulas for the commutator and the inner product of (q, p) and (q', p') in $\mathbf{O} \times \mathbf{O}_0$ as those given by (1) and (2) above.

Writing N^- as the manifold $\mathfrak{n}^-$ with the group multiplication given by the Campbell-Hausdorff formula we obtain

PROPOSITION 1. *The underlying manifold for the nilpotent group N^- is $\mathbf{F}^{r-1} \times \mathbf{F}_0$ with $r \geqslant 2$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ ($\mathbf{F}_0 = \{0\}$ if $\mathbf{F} = \mathbf{R}$) and with $r = 2$ for $\mathbf{F} = \mathbf{O}$. The group law is*

$$(q, p)(q', p') = (q + q', p + p' + \mathrm{Im}(\bar{q} \cdot q')).$$

The Haar measure on N^- is the ordinary Lebesgue measure on $\mathbf{R}^k \approx \mathbf{F}^{r-1} \times \mathbf{F}_0$, $k = r\sigma - 1$. We normalize it so that the volume of the unit cube in $\mathbf{R}^k$ is 1 and denote by $dqdp$. The Poisson kernel is given by

$$P_{\exp(tH)}(q, p) = c_{r,\mathbf{F}} \varepsilon^{d/2} \left[(|q|^2 + \varepsilon)^2 + 4|p|^2 \right]^{-d/2},$$

where $\varepsilon = e^{-t}$, $d = (r+1)\sigma - 2$, $|q|^2 = \bar{q} \cdot q$, $c_{r,\mathbf{F}} = 2^{d-1}\pi^{-rs}\Gamma(rs)$ with $\sigma = 2s = \dim_{\mathbf{R}} \mathbf{F}$.

2. Holomorphically induced representations of N^-. The adjoint and coadjoint action of N^- on $\mathfrak{n}^-$ and $\mathfrak{n}^{-*}$, respectively, is given by

$$\mathrm{Ad}_{(q,p)}(q'', p'') = (q'', p'' + 2\,\mathrm{Im}(\bar{q} \cdot q'')),$$
$$\mathrm{Ad}^*_{(q,p)}(q', p') = (q' + 2q\bar{p}', p'),$$

$(q, p) \in N^-$, $(q'', p'') \in \mathfrak{n}^-$, $(q', p') \in \mathfrak{n}^{-*}$, $q\bar{p}' = (q_1\bar{p}', \ldots, q_{r-1}\bar{p}')$, and we have identified n^{-*}, the dual space of $\mathfrak{n}^-$, with $\mathfrak{n}^-$ by $\langle \cdot, \cdot \rangle = (\cdot, \cdot)_\theta / 4(m_\alpha + 4m_{2\alpha})$. The single points $(q', 0) \in \mathfrak{n}^{-*}$ are 0-dimensional orbits of Ad^* on $\mathfrak{n}^{-*}$ and the corresponding (1-dimensional) representations of N^- are given by the characters

$$\chi_{(q',0)}(q, p) = \exp\left(\sqrt{-1}\;\mathrm{Re}(\bar{q}' \cdot q)\right), \qquad (q, p) \in N^-.$$

The remaining (maximal dimensional) orbits, for $\mathbf{F} = \mathbf{C}, \mathbf{H}$ and $\mathbf{O}$, are of the form $\mathbf{F}^{r-1} \times \{p''\}$, $p'' \neq 0$, so they are parameterized, e.g., by the functionals $f = (0, p'') \in \mathfrak{n}^{-*}$ with $p'' \in F_0 \setminus \{0\}$. For such f and $(q, p), (q', p') \in \mathfrak{n}^-$ we have

$$\langle f, [(q, p), (q', p')] \rangle = 2\,\mathrm{Re}((qp'')^- \cdot q') = -2\,\mathrm{Re}(\bar{q} \cdot (q'p'')),$$

i.e. the operator $R_{p''}$: $q \to qp''$ is skew-symmetric on $\mathbf{F}^{r-1}$ with respect to the $\mathbf{R}$-bilinear symmetric form $\langle \cdot, \cdot \rangle$ on $\mathbf{F}^{r-1}$ given by $\langle q, q' \rangle = 2\,\mathrm{Re}(\bar{q} \cdot q')$ = the usual inner product on $\mathbf{R}^k$, $k = (r-1)\sigma$. Let 1, i denote the usual basis of $\mathbf{C}$ (over $\mathbf{R}$); 1, i, j, k the basis of $\mathbf{H}$, and 1, i, j, k, e, ie, je, ke the basis of $\mathbf{O}$ (= $\mathbf{H} + \mathbf{H}e$ with the multiplication defined by $(ae)b = (a\bar{b})e$, $a(be) = (ba)e$, $(ae)(be) = -\bar{b}a$ for $a, b \in \mathbf{H}$). Suppose now that $f = (0, i\lambda) \in \mathfrak{n}^{-*}$ with λ positive real. In the above bases of $\mathbf{F}$ the matrix of $R_{i\lambda}$ acting on $\mathbf{F}^{r-1}$ with $r = 2$ is equal to $\lambda(E_{21} - E_{12})$ for $\mathbf{F} = \mathbf{C}$, to $\lambda(E_{21} - E_{12}) - \lambda(E_{43} - E_{34})$ for $\mathbf{F} = \mathbf{H}$ and to $\lambda(E_{21} - E_{12}) - \lambda(E_{43} - E_{34}) - \lambda(E_{65} - E_{56}) + \lambda(E_{87} - E_{78})$ for $\mathbf{F} = \mathbf{O}$. Put

$$e_1 = \tfrac{1}{2}(1 + \sqrt{-1}\, i), \qquad e_2 = \tfrac{1}{2}(\sqrt{-1}\, j + k),$$
$$e_3 = \tfrac{1}{2}(\sqrt{-1}\, e + ie), \qquad e_4 = \tfrac{1}{2}(je + \sqrt{-1}\, ke)$$

for elements in $\mathbf{F}^{\mathbf{C}} = \mathbf{F} + \sqrt{-1}\,\mathbf{F}$–the complexification of $\mathbf{F}$. Now define a subspace W of $\mathbf{F}^{\mathbf{C}}$ by

$$W = \begin{cases} \mathbf{C}e_1 & \text{if } \mathbf{F} = \mathbf{C}, \\ \mathbf{C}e_1 + \mathbf{C}e_2 & \text{if } \mathbf{F} = \mathbf{H}, \\ \mathbf{C}e_1 + \mathbf{C}e_2 + \mathbf{C}e_3 + \mathbf{C}e_4 & \text{if } \mathbf{F} = \mathbf{O}. \end{cases}$$

Thus

$$\mathfrak{h} = W^{r-1} \times \mathbf{F}_0^{\mathbf{C}} \tag{2.1}$$

is a positive polarization at $f = (0, i\lambda)$ such that

$$\mathfrak{h} + \bar{\mathfrak{h}} = \mathfrak{n}^{-\mathbf{C}}, \quad \mathfrak{h} \cap \bar{\mathfrak{h}} = \{0\} \times \mathbf{F}_0^{\mathbf{C}}, \quad \mathfrak{h}/\{0\} \times \mathbf{F}_0^{\mathbf{C}} = W, \tag{2.2}$$

where $\bar{z} = x - \sqrt{-1}\, y$ for $z = x + \sqrt{-1}\, y \in \mathfrak{n}^- + \sqrt{-1}\,\mathfrak{n}^- = \mathfrak{n}^{-\mathbf{C}}$. For arbitrary $f = (0, p'')$ with $p'' \in \mathbf{F}_0 \times \{0\}$, there exists an orthogonal transformation Ω on $\mathbf{R}^\sigma \approx \mathbf{F}$, such that

$$\langle R_{p''}q, q' \rangle = \langle R_{i|p''|}{}^t\Omega q, {}^t\Omega q' \rangle, \qquad q, q' \in \mathbf{F}^{r-1}.$$

Hence,

$$\langle f, [(q,p),(q',p')] \rangle = \langle (0, i|p''|), [({}^t\Omega q, p), ({}^t\Omega q', p')] \rangle,$$

and $\mathfrak{h}' = \Omega\mathfrak{h}$, with $\mathfrak{h}$ as in (2.1), is a positive polarization at f and $\mathfrak{h}'$ satisfies (2.2) with $W' = \Omega W$ instead of W. Here $\Omega(q + \sqrt{-1}\, q')$ is understood as $\Omega q + \sqrt{-1}\,\Omega q'$, $q, q' \in \mathbf{F}^{r-1}$, and $\Omega q = (\Omega q_1, \ldots, \Omega q_{r-1}) \in \mathbf{F}^{r-1}$. As in [**1**, pp. 158–162] one obtains that the space $\mathcal{H}(f, \mathfrak{h})$ of the representation $\rho(f, \mathfrak{h})$ corresponding to the chosen f and $\mathfrak{h}'$ may be realized as a space of complex C^∞ functions ψ on the complex space $\overline{W'}$, square integrable with respect to the measure $\exp(-\sqrt{-1}\,\langle f, [Y, \bar{Y}] \rangle)\, dY d\bar{Y}$ ($dYd\bar{Y}$ denoting the Lebesgue measure) on $\overline{W'}$ and satisfying the following functional equation:

$$[\tau(\sqrt{-1}\,\bar{X})\psi](\bar{Y}) = \sqrt{-1}\,[\tau(\bar{X})\psi](\bar{Y}), \qquad X, Y \in W',$$

where $[\tau(\bar{X})\psi](\bar{Y}) = (d/dt)\psi(\bar{Y} + t\bar{X})|_{t=0}$. The representation ρ is given by

$$\left[\rho(f, \mathfrak{h})\left(\exp\left(\bar{X}_0 + X_0 + Z_0\right)\right)\psi\right](\bar{X})$$
$$= \exp\left(\sqrt{-1}\,\left\langle f, \left[X_0, \bar{X}\right]\right\rangle - (\sqrt{-1}\,/2)\left\langle f, \left[X_0, \bar{X}_0\right]\right\rangle\right)\chi_f(Z_0)\cdot\psi\left(\bar{X} - \bar{X}_0\right), \tag{2.3}$$

where $\bar{X}_0 + X_0 + Z_0$ is in $\mathfrak{n}^-$ for $X_0 \in W'$, $Z_0 \in \mathbf{F}_0$; $\chi_f(Z_0) = \exp(\sqrt{-1}\,\langle f, Z_0\rangle)$. Passing to the complex coordinates $(z_1, \ldots, z_{s(r-1)})$ on $\bar{W}'$, according to the identifications

$$\bar{X} = (z_1\bar{e}_1, \ldots, z_{r-1}\bar{e}_1) \leftrightarrow (z_1, \ldots, z_{r-1}) \quad \text{for } \mathbf{F} = \mathbf{C},$$
$$\bar{X} = (z_1\bar{e}_1 + z_2\bar{e}_2, \ldots, z_{2(r-1)-1}\bar{e}_1 + z_{2(r-1)}\bar{e}_2) \leftrightarrow (z_1, \ldots, z_{2(r-1)}) \quad \text{for } \mathbf{F} = \mathbf{H},$$
$$\bar{X} = (z_1\bar{e}_1 + z_2\bar{e}_2 + z_3\bar{e}_3 + z_4\bar{e}_4) \leftrightarrow (z_1, \ldots, z_4) \quad \text{for } \mathbf{F} = \mathbf{O},$$

we have

$$\sqrt{-1}\,\left\langle f, \left[X, \bar{X}'\right]\right\rangle = |p''|\bar{z}\cdot z', \quad X = (\bar{X})^-, \qquad \bar{X}, \bar{X}' \in \bar{W}',$$
$$f = (0, p''), \qquad z = (z_1, \ldots, z_{s(r-1)}) \in \mathbf{C}^{s(r-1)}.$$

Rewriting (2.3) in these coordinates we obtain

PROPOSITION 2.1. *All the inequivalent irreducible unitary representations of N^- fall into two classes*:

(a) *a family of* 1-*dimensional characters $\chi_{q'}$ parameterized by $q' \in \mathbf{F}^{r-1}$ and given by*

$$\chi_{q'}(q, p) = \exp\left(\sqrt{-1}\ \operatorname{Re}(\bar{q}'\cdot q)\right), \qquad (q, p) \in N^-;$$

(b) *a family of infinite-dimensional representations $\rho_{p'}$ parameterized by $p' \in F_0 \setminus \{0\}$. The Hilbert space $\mathcal{H}_{p'}$ of the representation $\rho_{p'}$ consists of holomorphic functions ψ on $\mathbf{C}^{s(r-1)}$, such that*

$$\|\psi\|_{p'}^2 = \int_{\mathbf{C}^{s(r-1)}} |\psi(z)|^2 \exp\left(-|p'|\,|z|^2\right) dz\, d\bar{z} < \infty,$$

with

$$dz\, d\bar{z} = \prod_{j=1}^{s(r-1)} 2d\operatorname{Re} z_j d\, Im\, z_j.$$

The action of $\rho_{p'}$ on $\psi \in \mathcal{H}_{p'}$ is given by

$$\left(\rho_{p'}(q, p)\psi\right)(z) = \exp\left(\sqrt{-1}\ \operatorname{Re}(\bar{p}'p) + |p'|\left(\bar{z}_0\cdot z - \tfrac{1}{2}|z_0|^2\right)\right)\psi(z - z_0),$$

$z \in \mathbf{C}^{s(r-1)}$, $(q, p) \in N^-$ with $q = q(z_0)$, $z_0 \in \mathbf{C}^{s(r-1)}$, where

$$q(z) = \Omega\left(P(z_1, \ldots, z_s), \ldots, P\left(z_{s(r-2)+1}, \ldots, z_{s(r-1)}\right)\right)$$

and $P(z_{s(l-1)+1}, \ldots, z_{sl})$, $l = 1, \ldots, r-1$, is defined as

$$\operatorname{Re} z_l + i\, Im\, z_l \quad \textit{for } \mathbf{F} = \mathbf{C},$$
$$\operatorname{Re} z_{2l-1} + i\, Im\, z_{2l-1} + j\, Im\, z_{2l} + k\operatorname{Re} z_{2l} \quad \textit{for } \mathbf{F} = \mathbf{H},$$
$$\operatorname{Re} z_1 + i\, Im\, z_1 + j\, Im\, z_2 + k\operatorname{Re} z_2 + e\, Im\, z_3$$
$$+ ie\operatorname{Re} z_3 + je\operatorname{Re} z_4 + ke\, Im\, z_4 \quad \textit{for } \mathbf{F} = \mathbf{O};$$

$Im(a + \sqrt{-1}\, b) = b$, $a, b \in \mathbf{R}$. The functions

$$\psi_n^{p'}(z) = (2\pi)^{-m/2}|p'|^{(|n|+m)/2}(n!)^{-1/2}z^n, \qquad z \in \mathbf{C}^m,$$

$n = (n_1, \ldots, n_m) \in \mathbf{N}^m$, with $n! = n_1! \cdots n_m!$, $z^n = z_1^{n_1} \cdots z_m^{n_m}$, $|n| = n_1 + \cdots + n_m$, $m = s(r-1)$, form an orthonormal basis of $\mathcal{H}_{p'}$ as n runs over $\mathbf{N}^m$.

We also note the following symmetry properties of $\chi_{q'}$ and $\rho_{p'}$ relative to the orthogonal and the unitary transformations.

PROPOSITION 2.2. (a) *Let $o_1, \ldots, o_{r-1} \in O(\sigma, \mathbf{R})$; then for $q' \in \mathbf{F}^{r-1}$,*

$$\chi_{q'}({}^t o_1 q_1, \ldots, {}^t o_{r-1} q_{r-1}, p) = \chi_{(o_1 q'_1, \ldots, o_{r-1} q'_{r-1})}(q, p), \qquad (q, p) \in N^-.$$

(b) *Let $u_1, \ldots, u_{r-1} \in U(s)$; then for $p' \in F_0 \setminus \{0\}$,*

$$A_u^{-1}\rho_{p'}(q, p)A_u = \rho_{p'}(q^u, p),$$

$(q, p) \in N^-$ with $q = q(z_0)$, $z_0 \in \mathbf{C}^{s(r-1)}$ and $q^u = q(uz_0)$ with $uz = (u_1(z_1, \ldots, z_s), \ldots, u_{r-1}(z_{s(r-2)+1}, \ldots, z_{s(r-1)}))$; $((A_u)\psi)(z) = \psi(uz)$ for $\psi \in \mathcal{H}_{p'}$; $u = (u_1, \ldots, u_{r-1})$, $\sigma = 2s = \dim_{\mathbf{R}} \mathbf{F}$.

3. Algebra of multiradial functions.

DEFINITION (CF. GELLER [2]). We say that a function F on $N^- = \mathbf{F}^{r-1} \times \mathbf{F}_0$, $\mathbf{F} = \mathbf{C}, \mathbf{H}, \mathbf{O}$, is *multiradial* if there is a function f on $\mathbf{R}_+^{r-1} \times \mathbf{F}_0$ such that

$$F(q, p) = f(|q_1|, \ldots, |q_{r-1}|, p), \qquad (q, p) \in N^-. \tag{3.0}$$

PROPOSITION 3.1. *Let $\mathcal{Q}$ denote the space of multiradial functions in $L^1(N^-)$. Then $\mathcal{Q}$ is a commutative closed *-subalgebra of $L^1(N^-)$ and $\mathcal{Q}$ is symmetric.*

PROOF. 1°. If $F, G \in \mathcal{Q}$ then $F * G \in \mathcal{Q}$. For we have

$$F * G(q', p') = \int f(|q_1|, \ldots, |q_{r-1}|, p) \times g(|q'_1 - q_1|, \ldots, |q'_{r-1} - q_{r-1}|, p' - p - \operatorname{Im}(\bar{q} \cdot q'))\, dq\, dp. \tag{3.1}$$

Substituting $q = ((q'_1/|q'_1|)\tilde{q}_1, \ldots, (q'_{r-1}/|q'_{r-1}|)\tilde{q}_{r-1})$ we get (since $(a\bar{b})b = a(\bar{b}b)$ for $a, b \in \mathbf{F}$)

$$\int f(|\tilde{q}_1|, \ldots, |\tilde{q}_{r-1}|, p) g\Big(|q'_1|\, |1 - \tilde{q}_1/|q'_1||, \ldots, |q'_{r-1}|\, |1 - \tilde{q}_{r-1}/|q'_{r-1}||, p' - p - \operatorname{Im}\Big(\sum_{l=1}^{r-1} \bar{\tilde{q}}_l |q'_l|\Big)\Big)\, d\tilde{q}\, dp,$$

i.e. $F * G$ is multiradial. Obviously $\mathcal{Q}$ is closed.

2°. $\mathcal{Q}$ is commutative (cf. Kaplan and Putz [6, p. 377]). Under the orthogonal change of variables

$$q_l \mapsto q''_l = q'_l \cdot 2\operatorname{Re}(\bar{q}_l q'_l)/|q'_l|^2 - q_l, \qquad l = 1, \ldots, r-1,$$

one has $|q_l' - q_l''| = |q_l' - q_l|$ and $\mathrm{Im}(\bar{q}_l q_l') = -\mathrm{Im}(\bar{q}_l'' q_l')$. Thus (3.1) is equal to

$$\int f(|q_1''|, \ldots, |q_{r-1}''|, p)$$

$$\times\, g\left(|q_1' - q_1''|, \ldots, |q_{r-1}' - q_{r-1}''|, p' - p - \sum_{l=1}^{r-1} - \mathrm{Im}(\bar{q}_l'' q_l')\right) dq''\, dp$$

$$= \int_{N^-} F(q'', p) G\big((q', p')(q'', p)^{-1}\big)\, dq''\, dp$$

$$= G * F(q', p').$$

3°. Since $L^1(N^-)$ is symmetric (Leptin [**8**, p. 205]), its *-subalgebra $\mathcal{Q}$ is also symmetric.

Proposition 3.2. *For $F \in \mathcal{Q}$ and $u = (u_1, \ldots, u_{r-1}) \in U(s) \times \cdots \times U(s)$, the operators $\rho_{p'}(F) = \int_{N^-} F(q, p)\rho_{p'}(q, p)\, dq\, dp$ and A_u commute on $\mathcal{H}_{p'}$.*

Proof. By Proposition 2.2(b),

$$A_u^{-1} \int \rho_{p'}(q(z_0), p) F(q(z_0), p)\, dq(z_0)\, dp\, A_u$$

$$= \int \rho_{p'}(q(uz_0), p) F(q(z_0), p)\, dq(z_0)\, dp.$$

Since

$$q_l(uz_0) = \Omega(P(u_l Z_l)) = \big(\Omega P u_l P^{-1}\Omega^{-1}\big)(\Omega P Z_l)$$

with $Z_l = (z^0_{s(l-1)+1}, \ldots, z^0_{sl})$, $l = 1, \ldots, r-1$, and since $\Omega P u_l P^{-1}\Omega^{-1}$ is an orthogonal transformation on $\mathbf{R}^\sigma \approx \mathbf{F}$, and $\Omega P Z_l = q_l(z_0)$, the last integral is equal to $\rho_{p'}(F)$.

Remark. For $\mathbf{F} = \mathbf{R}$, the corresponding group N^- is $\mathbf{R}^{r-1}$, so the algebra $L^1(N^-)$ is already commutative, and, as in the case $M = \mathbf{R}^n \times \mathbf{R}_+$ with the Euclidean metric [**5**], we consider $\mathcal{Q} = L^1(N^-)$.

4. Multiplicative linear functionals on $\mathcal{Q}$. Let Φ be a nonzero multiplicative linear functional on $\mathcal{Q}$. Since $\mathcal{Q}$ is a symmetric *-subalgebra of $L^1(N^-)$, there exist an irreducible *-representation π of $L^1(N^-)$ and a unit vector ξ in the Hilbert space $\mathcal{H}_\pi$ such that

$$\pi(F)\xi = \Phi(F)\xi \quad \text{for all } F \text{ in } \mathcal{Q}. \tag{4.1}$$

If $\mathcal{H}_\pi$ is one dimensional, then

$$\pi(F)\xi = \int_{N^-} F(q, p)\chi_{q'}(q, p)\, dq\, dp\xi \tag{4.2}$$

for some $q' \in \mathbf{F}^{r-1}$, and by Proposition 2.2(a), if q' and q'' in $\mathbf{F}^{r-1}$ are such that $|q_l'| = |q_l''|$, $l = 1, \ldots, r-1$, the Φ's corresponding by (4.1) and (4.2) to $\chi_{q'}$ and $\chi_{q''}$ are identical. If $\pi \approx \rho_{p'}$, then by Proposition 3.2, $\rho_{p'}(F)$ and A_u commute. Now for $\psi(z) = \psi_1(Z_1) \ldots \psi_{r-1}(Z_{r-1})$ with $z = (Z_1, \ldots, Z_{r-1})$, $Z_l = (z_{s(l-1)+1}, \ldots, z_{sl})$, $l = 1, \ldots, r-1$, we have

$$(A_u\psi)(z) = \psi_1(u_1 Z_1) \ldots \psi_{r-1}(u_{r-1} Z_{r-1}).$$

Thus putting $\psi_l(Z_l) = Z_l^{n_l}$ with $n_l = (n_{1l}, \ldots, n_{sl}) \in \mathbf{N}^s$, we note that A_u preserves the finite-dimensional subspaces of $\mathcal{H}_{p'}$, namely the spaces $\mathcal{H}^n = \otimes_{l=1}^{r-1} \mathcal{H}^{|n_l|}$, where $n = (|n_1|, \ldots, |n_{r-1}|) \in \mathbf{N}^{r-1}$. $\mathcal{H}^{|n_l|}$ is the space of homogeneous polynomials in $z_{s(l-1)}, \ldots, z_{sl}$ of degree $|n_l|$. Moreover, $\mathcal{H}_{p'} = \oplus_n \mathcal{H}^n$–an orthogonal direct sum over $n \in \mathbf{N}^{r-1}$. We also note that A_u restricted to $\mathcal{H}^n$ is equal to $\otimes_{l=1}^{r-1} T^{|n_l|}(u_1^{-1}, \ldots, u_{r-1}^{-1})$ with T^k, $k = |n_l|$, being the representation of $U(s)$ on $\mathcal{H}^k$ given by $(T_u^k\psi)(Z) = \psi(u^{-1}Z)$. Since T^k is irreducible (cf., e.g., **[13**, pp. 204–209**]**), the representations $T^n = \otimes_l T^{|n_l|}$ of $U(s) \times \cdots \times U(s)$, $r - 1$ copies of $U(s)$, act irreducibly on $\mathcal{H}^n$, and $T^n \approx T^m$ iff $n = m$. Hence, by Schur's Lemma, every intertwining operator S for $\oplus_n T^n$ on $\mathcal{H}_{p'}$ is of the form $S = \oplus_n c_n(S)\mathrm{Id}_{\mathcal{H}^n}$. In particular, each $\rho_{p'}(F)$ with $F \in \mathcal{Q}$ is such. It follows from (4.1) that $\Phi(F)$ is equal to one of the constants $c_n(\rho_{p'}(F))$, $n \in \mathbf{N}^{r-1}$. Conversely, for every fixed n, the mapping $F \mapsto c_n(\rho_{p'}(F))$ defines a multiplicative linear functional on $\mathcal{Q}$. Now we shall derive explicit formulas for the constants c_n above. Since, e.g.,

$$c_n(\rho_{p'}(F)) = (\rho_{p'}(F)\psi_{n'}^{p'}, \psi_{n'}^{p'})_{\mathcal{H}_{p'}}$$

with $n' = (n_1, 0, \ldots, 0; n_2, 0, \ldots, 0; \ldots; n_{r-1}, 0, \ldots, 0) \in (\mathbf{N}^s)^{r-1}$, we calculate the integral, see Proposition 2.1(b),

$$\int_{\mathbf{C}^{s(r-1)}} [\rho_{p'}(F)\psi_{n'}^{p'}](z)\bar{\psi}_{n'}^{p'}(z)\exp(-|p'|\,|z|^2)\,dz\,d\bar{z}, \tag{4.3}$$

which in expanded form is equal to (with $k = s(r-1)$)

$$\begin{aligned}(2\pi)^{-k}(n!)^{-1}|p'|^{|n|+k}\\ \times \int_{\mathbf{C}^k}\int_{\mathbf{F}^{r-1}\times\mathbf{F}_0} F(q(z_0), p)\exp\left(\sqrt{-1}\ \mathrm{Re}(\bar{p}'p) + |p'|\left(\bar{z}_0 \cdot z - \tfrac{1}{2}|z_0|^2\right)\right)\\ \times (z - z_0)^{n'}\bar{z}^{n'}\exp(-|p'|\,|z|^2)\,dq(z_0)\,dp\,dz\,d\bar{z}.\end{aligned} \tag{4.4}$$

The integral

$$\int_{\mathbf{C}^{s(r-1)}} (z - z_0)^{n'}\bar{z}^{n'}\exp(-|p'|\,|z|^2)\exp(|p'|\bar{z}_0 \cdot z)\,dz\,d\bar{z} \tag{4.5}$$

is equal to

$$\begin{aligned}(2\pi/|p'|)^{(s-1)(r-1)}\prod_{l=1}^{r-1} 2\pi n_l!|p'|^{-n_l-1}\sum_{j=0}^{n_l}\left(-|p'|\,|z^0_{s(l-1)+1}|^2\right)^j\binom{n_l}{j}(j!)^{-1}\\ = (2\pi)^{s(r-1)}|p'|^{-|n|-s(r-1)}n!\prod_{l=1}^{r-1} L_{n_l}\left(|p'|\,|z^0_{s(l-1)+1}|^2\right)\end{aligned} \tag{4.5a}$$

with L_{n_l} being the Laguerre polynomial. (4.5a) is obtained (see **[5]**) by substituting the binomial formula for $(z - z_0)^n$, developing $\exp(-|p'|\bar{z}_0 \cdot z)$ in a power series and integrating this series term by term using the orthogonality relations for the functions z^n in $\mathcal{H}_{p'}$. Substituting (4.5a) in (4.4) we obtain that (4.3) is equal to

$$\int_{\mathbf{F}^{r-1}\times\mathbf{F}_0} F(q(z_0), p)\exp\left(\sqrt{-1}\ \operatorname{Re}(\bar{p}'p) - \tfrac{1}{2}|p'|\,|z_0|^2\right)$$

$$\times \prod_{l=1}^{r-1} L_{n_l}\left(|p'|\,\left|z^0_{s(l-1)+1}\right|^2\right) dq(z_0)\, dp$$

$$= \int_0^\infty dt_1 \ldots \int_0^\infty dt_{r-1}\left(\int_{\mathbf{F}_0} f(t_1, \ldots, t_{r-1}, p)\exp\left(\sqrt{-1}\ \operatorname{Re}(\bar{p}'p)\right) dp\right)$$

$$\times \exp\left(-\tfrac{1}{2}|p'|(t_1^2 + \ldots + t_{r-1}^2)\right) \prod_{l=1}^{r-1} t_l^{\sigma-1}\mathcal{G}_l,$$

with f as in (3.0) and $\mathcal{G}_l$ given by

$$\mathcal{G}_l = \int_{S(\sigma-1)} L_{n_l}\left(|p'|\,|z_1|^2\right) dS(q(Z)),$$

$$Z = (z_1, \ldots, z_s) \in \mathbf{C}^s, \qquad |Z| = t_l,$$

$S(\sigma - 1)$ being the unit sphere in $\mathbf{F}$. Since here $q(Z) = \Omega(PZ)/t_l$, with P as in Proposition 2.1(b) and $\Omega \in O(\sigma, \mathbf{R})$, in order to compute $\mathcal{G}_l$ one has to calculate the integrals

$$\int_{S(\sigma-1)} |z_1(q)|^{2j}\, dS(q), \qquad j = 0, \ldots, n_l, \tag{4.6}$$

where $|z_1(q)|^2 = t_l^2((q^1)^2 + (q^i)^2)$ with $q^1, q^i, \ldots$ denoting the coordinates of q in the (standard) basis $\{1, i, \ldots\}$ of $\mathbf{F}$ over $\mathbf{R}$. Now (4.6) is equal to

$$\int_0^{\pi/2} \cos^{2j}\theta \cos\theta \sin^{\sigma-3}\theta\, d\theta \cdot 2\pi \cdot 2\pi^{s-1}\left[(s-2)!\right]^{-1} t_l^{2j}$$

$$= 2\pi^s t_l^{2j} j!/(j+s-1)!.$$

We summarize the results of this section in the following:

PROPOSITION 4. *The multiplicative linear functionals on $\mathcal{Q}$ fall into two classes*:

(a) *the functionals corresponding to $(r-1)$-tuples $(t_1, \ldots, t_{r-1})$ of nonnegative real numbers and given by*

$$F \mapsto \hat{F}(t_1, \ldots, t_{r-1}) = \int_{\mathbf{F}^{r-1}\times\mathbf{F}_0} F(q, p)\exp\left(\sqrt{-1}\ \operatorname{Re}(\bar{q}'\cdot q)\right) dq\, dp$$

with $q' \in \mathbf{F}^{r-1}$ arbitrary provided $(|q'_1|, \ldots, |q'_{r-1}|) = (t_1, \ldots, t_{r-1})$.

(b) *the functionals corresponding to pairs $(p', n) \in \mathbf{F}_0 \setminus \{0\} \times \mathbf{N}^{r-1}$ and given by*

$$F \mapsto \hat{F}(p', n) = (2\pi^s)^{r-1}\int_{\mathbf{R}_+^{r-1}} \exp\left(-\frac{|p'|}{2}(t_1^2 + \cdots + t_{r-1}^2)\right)\prod_{l=1}^{r-1} L_{n_l}^{(s-1)}\left(|p'|t_l^2\right)t_l^{\sigma-1}$$

$$\times\left(\int_{\mathbf{F}_0} f(t_1, \ldots, t_{r-1}, p)\exp\left(\sqrt{-1}\ \operatorname{Re}(\bar{p}'p)\right) dp\right) dt_1 \ldots dt_{r-1},$$

where $f(|q_1|, \ldots, |q_{r-1}|, p) = F(q_1, \ldots, q_{r-1}, p)$ *and*

$$L_k^{(m)}(x) = \sum_{j=0}^{k} \frac{(-x)^j}{(j+m)!}\binom{k}{j}$$
$$= \left[(k+m)!\right]^{-1} x^{-m} e^{x} (d^k/dx^k)(x^{k+m}e^{-x}). \tag{4.7}$$

5. Nonvanishing of the Gel'fand transform of P_a.

LEMMA 1. *For* $1 \leqslant m < 2k + \frac{1}{2}$, $k > \frac{3}{2}$ *and* $Q > 0$, *the following formula holds*:

$$\int_{\mathbf{R}^m} \frac{\exp(\sqrt{-1}\; x_0 \cdot x)\, dx}{(Q^2 + 4|x|^2)^k} = 2^{-m}\pi^{m/2} \frac{\Gamma(k - m/2)}{\Gamma(k)} Q^{m-2k}, \quad \textit{for } x_0 = 0,$$
$$= 2^{m+1-4k}\pi^{(m+1)/2} \frac{r^{2k-m}e^{-(r/2)Q}}{\Gamma(k)\Gamma(k - (m-1)/2)}$$
$$\times \int_0^\infty e^{-(r/2)Qt}\left((t+1)^2 - 1\right)^{k-(m+1)/2} dt,$$
$$\textit{for } r = |x_0| \neq 0. \tag{5.1}$$

PROOF. For $x_0 = 0$, the integral is equal to the "area" of the unit sphere in $\mathbf{R}^m$ (= 2 when $m = 1$) times $\int_{\mathbf{R}_+} r^{m-1}(Q^2 + 4r^2)^{-k}\, dr$ and we substitute $r = r'Q/2$. For $x_0 \neq 0$, the function $[4((\frac{1}{2}Q)^2 + x^2)]^{-k}$ is radial on $\mathbf{R}^m$, hence its Fourier transform (5.1) is equal to, see, e.g., [**9**, p. 155],

$$4^{-k}(2\pi)^{m/2} r^{-(m-2)/2} \int_0^\infty \left(\left(\frac{1}{2}Q\right)^2 + t^2\right)^{-k} J_{(m-2)/2}(rt)\, t^{m/2}\, dt,$$
$$m \geqslant 1, \qquad k > 3/2.$$

Combining now the Sonine formula [**12**, p. 434, (2)],

$$\int_0^\infty \frac{x^{\nu+1}J_\nu(ax)\, dx}{(x^2 + k^2)^{\mu+1}} = \frac{a^\mu k^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} K_{\mu-\nu}(ak),$$

valid when $-1 < \operatorname{Re}(\nu) < 2\operatorname{Re}(\mu) + \frac{3}{2}$, with the following expression for the function K [**12**, p. 172, (4)],

$$K_\nu(z) = \frac{\Gamma(\frac{1}{2})(\frac{1}{2}z)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\nu - 1/2}\, dt,$$

valid for $\operatorname{Re}(\nu + \frac{1}{2}) > 0$, $|\arg z| < \pi/2$, we obtain (5.1).

LEMMA 2. *For* $\varepsilon > 0$, $m > 0$, $x \in \mathbf{R}^n$,

$$(\varepsilon + |x|^2)^{-m} = (4\pi)^{-n/2}\Gamma(m)^{-1} \int_{\mathbf{R}^n} \exp(-\sqrt{-1}\; x \cdot y)$$
$$\times \left(\int_0^\infty t^{m-1-m/2} e^{-\varepsilon t} e^{-|y|^2/4t}\, dt\right) dy,$$

i.e. $(\varepsilon + |x|^2)^{-m}$ *is a Fourier transform of a positive function in* $L^1(\mathbf{R}^n)$.

PROOF. Combine

$$\left(\varepsilon+|x|^2\right)^{-m}=\Gamma(m)^{-1}\int_0^\infty t^{m-1}e^{-(\varepsilon+|x|^2)t}\,dt,\qquad m>0,$$

with

$$\exp\left(-|x|^2t\right)=(4\pi t)^{-n/2}\int_{\mathbf{R}^n}\exp\left(-|y|^2/4t\right)\exp\left(-\sqrt{-1}\;x\cdot y\right)dy,$$

and note that the obtained double integral is absolutely convergent.

PROPOSITION 5. *For every $a\in A$, $\hat{P}_a$ does not vanish on $\mathfrak{M}(\mathcal{Q})$–the maximal ideal space of $\mathcal{Q}$.*

PROOF. (a) For the points $(t_1,\dots,t_{r-1})\in\mathfrak{M}(\mathcal{Q})$, integrating over $\mathbf{F}_0$ in the formula of Proposition 4(a), according to Lemma 1 (with $x_0=0$) we get

$$\hat{P}_a(t_1,\dots,t_{r-1})=c\int_{\mathbf{R}^{\sigma(r-1)}}\frac{\exp\left(\sqrt{-1}\;\mathrm{Re}(\bar{q}'\cdot q)\right)}{\left(\varepsilon+|q|^2\right)^{d+1-\sigma}}\,dq,$$

with

$$c=2^{1-\sigma}\pi^{s-1/2}\varepsilon^{d/2}c_{r,\mathbf{F}}\Gamma(d/2-s+1/2)\Gamma(d/2)^{-1},$$

and this is positive by Lemma 2 and the Fourier inversion formula.

(b) For the points $(p',n)\in\mathfrak{M}(\mathcal{Q})$ with $p'\in\mathbf{F}_0\setminus\{0\}$, $n=(n_1,\dots,n_{r-1})\in\mathbf{N}^{r-1}$, we use the formula from Proposition 4(b) for $\hat{P}_a(p',n)$. Applying Lemma 1 to the integral over $\mathbf{F}_0$ there (with $Q=\varepsilon+t_1^2+\cdots+t_{r-1}^2$, $m=\sigma-1$, $x_0=p'$, $k=d/2$), then interchanging the order of integration from $dt\,dt_1\dots dt_{r-1}$ to $dt_1\dots dt_{r-1}dt$, making the change of variables $(|p'|t_1^2,\dots,|p'|t_{r-1}^2)=(x_1,\dots,x_{r-1})$, and finally applying (4.7), we obtain

$$\hat{P}_a(p',n)=c\int_0^\infty e^{-|p'|\varepsilon t/2}\left((t+1)^2-1\right)^{d/2-s}\prod_{l=1}^{r-1}\mathcal{G}_l(t)\,dt,$$

with

$$c=2^{s(r-1)}(\varepsilon|p'|)^{d/2}\Gamma(d/2)^{-1}\exp(-|p'|\varepsilon/2)$$

and

$$\mathcal{G}_l(t)=\left((n_l+s-1)!\right)^{-1}\int_0^\infty e^{-tx/2}(d^{n_l}/dx^{n_l})(x^{n_l+s-1}e^{-x})\,dx.$$

Integrating by parts get

$$\begin{aligned}\mathcal{G}_l(t)&=\left((n_l+s-1)!\right)^{-1}\int_0^\infty e^{-tx/2}x^{n_l+s-1}e^{-x}\,dx\cdot(t/2)^{n_l}\\&=(t/2)^{n_l}(t/2+1)^{-(n_l+s)}.\end{aligned}$$

Thus $\hat{P}_a(p',n)$ is positive.

REMARK. For M being the real hyperbolic space, i.e. for $\mathbf{F}=\mathbf{R}$, we have $N^-=\mathbf{R}^d$, $\mathfrak{M}(L^1(N^-))=\hat{N}^-$, $P_a(X_{-\alpha})=c_{d,\mathbf{R}}\varepsilon^{d/2}(\varepsilon+|q|^2)^{-d}$ and, by Lemma 2, $\hat{P}_a>0$ on $\hat{N}^-$.

6. Theorem on ideals in $L^1(N^-)$. Since the algebras $\mathcal{A}$ we consider here have the same qualitative properties as the one considered in [5], similar facts can be proved about them. In particular, the following statement about ideals in $L^1(N^-)$ is a consequence of the Wiener property of $\mathcal{A}$ and existence of the approximate identity for $L^1(N^-)$ in $\mathcal{A}$ (the dilations δ_t, $t > 0$, on N^- used in the construction of the approximate identity are given by $\delta_t(q, p) = (t^{-1/2}q, t^{-1}p)$).

PROPOSITION 6. *If $\mathcal{I}$ is a proper closed right ideal in $L^1(N^-)$, then there is a Φ in $\mathfrak{M}(\mathcal{A})$ such that $\hat{F}(\Phi) = 0$ for every $F \in \mathcal{I} \cap \mathcal{A}$.*

7. Proof of the Theorem [5]. The Theorem follows now from Proposition 6, for if we put

$$\mathcal{I} = \left\{ f \in L^1(N^-): \lim_{N^- \ni n \to \infty} \varphi * f(n) = c_0 \int_{N^-} f(n)\, dn \right\},$$

with $\varphi \in L^\infty(N^-)$ being the boundary value of the bounded harmonic function u on M, then $P_{a_0} \in \mathcal{I} \cap \mathcal{A}$ and $\hat{P}_{a_0} \neq 0$ on $\mathfrak{M}(\mathcal{A})$, so $\mathcal{I} = L^1(N^-)$. Hence $P_a \in \mathcal{I}$ for every a in A.

ADDED IN PROOF. Meanwhile Korányi [14] described the Gel'fand space, as well as the related Plancherel formula, for the commutative algebra $\mathcal{A}$ of *biradial* functions in $L^1(N^-)$, i.e. the functions F such that

$$F(q, p) = f(|q|, |p|), \qquad (q, p) \in N^-,$$

for some f on $\mathbf{R}_+ \times \mathbf{R}_+$, cf. §3. His approach uses neither the classification of symmetric spaces nor the representations of nilpotent groups.

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